Name:

CHAPTER 10

Lesson 45

Limits Revisited: *L'Hopital's Rule*

Chapter 10 deals with certain types of problems involving limits. The chapter contains problems with names that sound like they cannot be determined, as well as problems that don't know their manners and behave improperly. However, you should not be concerned. You will learn some very nice methods that will enable you to handle such unruly problems with ease.

We begin with a simple rule that was discovered over 300 years ago.

1. Consider these facts: Let c represent any real number other than zero.

a. $\lim\limits_{x \to 0} \dfrac{x}{c} =$ ____________________

b. $\lim\limits_{x \to 0} \dfrac{c}{x} =$ ____________________

2. For each example, make the substitution for x and see what form the rational expression takes.

a. $\lim\limits_{x \to 2} \dfrac{\sin(x-2)}{(x-2)} =$ ____________________

b. $\lim\limits_{x \to 0} \dfrac{\cos(x)-1}{x} =$ ____________________

c. $\lim\limits_{x \to 0} \dfrac{e^x-1}{x^2} =$ ____________________

d. $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2} =$ ____________________

3. Why are you not able to use substitution to find the limits in #2? ____

4. Use the TECH-TIP below to find the following limits. Write the answers in the spaces below.

TECH-TIP	**HOME Screen: F3 Calculus Menu: 3 limit(**	*page 330*
	limit($f(x)$, x, c) Options: limit($f(x)$, x, c, −1) limit($f(x)$, x, c, 1)	*Example*: limit(sin(x − 2)/(x − 2), x, 2) indicates the left-hand limit indicates the right-hand limit

a. $\lim\limits_{x \to 2} \dfrac{\sin(x-2)}{(x-2)} =$ ____________________

b. $\lim\limits_{x \to 0} \dfrac{\cos(x)-1}{x} =$ ____________________

c. $\lim\limits_{x \to 0^+} \dfrac{e^x-1}{x^2} =$ ____________________

d. $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2} =$ ____________________

5. The puzzling thing about the form $\frac{0}{0}$ is that its value is not always the same. Give four different values that you have found for $\lim_{x \to c} \frac{f(c)}{g(c)}$ when $f(c) = 0$ and $g(c) = 0$.

a. $\lim_{x \to c} \frac{0}{0} =$ ________________________

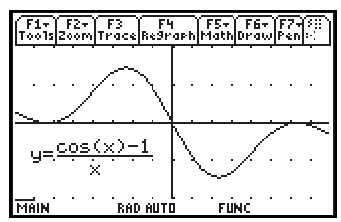
b. $\lim_{x \to c} \frac{0}{0} =$ ________________________

c. $\lim_{x \to c} \frac{0}{0} =$ ________________________

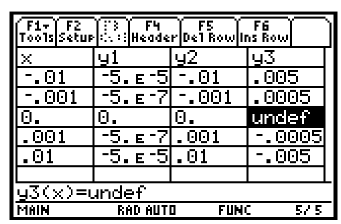
d. $\lim_{x \to c} \frac{0}{0} =$ ________________________

- The zero in the denominator usually makes the ratio approach infinity as in $\lim_{x \to c} \frac{c}{0} = \pm\infty$.
- The zero in the numerator usually makes the ratio approach zero as in $\lim_{x \to c} \frac{0}{c} = 0$.
- These two opposing tendencies mean that the limit of the form $\frac{0}{0}$ is **indeterminate**.
- The following are forms that are **indeterminate** having two opposing tendencies:

$\frac{0}{0}$ $\quad \frac{\infty}{\infty}$ $\quad \infty \cdot 0$ $\quad \infty - \infty$ $\quad 1^{\infty}$ $\quad 0^0$ $\quad \infty^0$



With the technology of today, you have graphical and numerical techniques that help you discover the limiting value; however, you should learn to be skeptical of such evidence. Even though these methods can give support for your answer, they are no substitute for an analytical justification.



x	y1	y2	y3
-.01	-5.E-5	-.01	.005
-.001	-5.E-7	-.001	.0005
0.	0.	0.	undef
.001	-5.E-7	.001	-.0005
.01	-5.E-5	.01	-.005

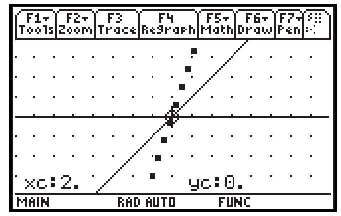
y3(x)=undef

At last you have reached the point in your calculus career where you need to understand how to handle these **indeterminate** forms. Yet the method is so direct and simple that it makes taking limits of many forms much easier. You will wonder why you weren't taught this little secret earlier. As you complete Chapters 10 and 11, you will grow to appreciate the full extent of its power.

The prolific writer **Leonhard Euler** called it "a known rule." The rule was first written in a letter to the French mathematician **Guillaume Francois Antoine de l'Hopital** (1661-1704). The wealthy Marquis de l'Hopital had paid the great Professor **Johann**, or John, **Bernoulli** (1667-1748) to keep him informed of the latest developments of the Calculus, which was in its infancy. In a letter dated 1694, Bernoulli sent his solution to the **indeterminate** problem. L'Hopital included it, without attribution, in his book published in 1696, the first textbook on differential calculus. Ironically, Bernoulli's simple but elegant solution became known as **l'Hopital's Rule**.

Follow the steps on the pages ahead to see the solution unfold.

Limits Revisited: *L'Hopital's Rule*



6. $\lim_{x\to c}\frac{f(c)}{g(c)} = \lim_{x\to 2}\frac{\sin(x-2)}{3(x-2)}$

Calculator Setup

- [◆] [F1] for [Y=]: [F1] Tools: [9] Graph Formats: Order – **Simultaneous**: Grid – **ON**: [ENTER]
- [F1] Tools: [8] Clear Functions: [ENTER] Define the following functions:
- Let y1 = $f(x) = \sin(x-2)$.
- Let y2 = $g(x) = 3(x - 2)$. [2nd] [F1], for [F6] **Style**, [3] **Square**.
- [F2] Zoom [4] Decimal Window.
- [F3] **Trace**: [2] [ENTER] to move to the point (2, 0).
- [F2] Zoom: [2] **ZoomIn**: [ENTER]
- Repeat zooming in, centered at $x = 2$ until the graph of y1 appears to be a straight line.

a. Record the graphing Window: ________________________

b. What property of a function guarantees that it will look like a line when you ZoomIn so that Δx is small? _______________

____

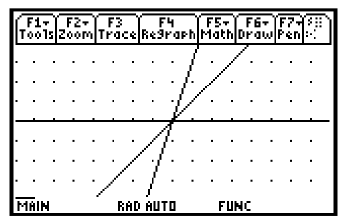
c. The graph of y2(x) is a line.

- Write the equation of the line that is tangent to the graph of y1(x) at the point (2, 0). ___________________________
- Enter the equation of the tangent line as y3(x) using [2nd] [F1], for [F6] **Style**: [6] **Path**.
- [◆] [F3]Graph. Press [F4] Regraph to see the three graphs again.

d. Perhaps this graph is what John Bernoulli saw in his mind's eye centuries ago. At the very least, we know that he understood the concept: "As x gets very close to the value of 2, the graphs of y1(x) $= f(x)$ and y2(x) $= g(x)$ get very close to **the graphs of their linear approximations.**"

- Translate that English sentence into a mathematical sentence.
- Graph y1, y2 and y3 in the Window: X [-0.5, 4.5, 0.5]; Y [-1.25, 1.25, 0.5]; xres = 2.
- Use that information in the context of problem 6.

$\lim_{x\to 2}\frac{\sin(x-2)}{3(x-2)} \quad \lim_{x\to 2}\frac{f(x)}{g(x)} \approx \lim_{x\to 2}$ ________________



e. Find the value of the new limit. Show your work.

(i) $f(2) = [\sin(x-2)]_{x=2}$ = ____________ $\qquad g(2) = [3(x-2)]_{x=2}$ = ____________

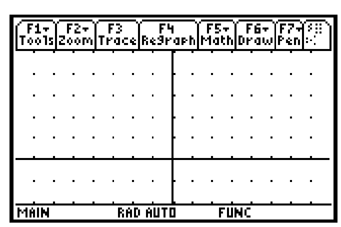
(ii) $\frac{d}{dx}[\sin(x-2)]_{x=2}$ = ____________ $\qquad \frac{d}{dx}[3(x-2)]_{x=2}$ = ____________

Simplify Algebraically:

$\lim_{x\to 2}\frac{f(x)}{g(x)} \approx \lim_{x\to 2}\frac{f(2)+f'(2)\cdot(x-2)}{g(2)+g'(2)\cdot(x-2)} = \lim_{x\to 2}$ ________________ = ____________________

f. Sketch the graph of the function $h(x) = \frac{\sin(x-2)}{3(x-2)}$ in the space provided.

- Use the Window: X [-7.9, 7.9, 1]; Y [-0.2, 0.5, 0.1].
- Draw a circle to indicate any hole(s) in the graph.
- What are the coordinates of the point(s)? ________________________________



Limits Revisited: *L'Hopital's Rule*

7. The solution to the problem $\frac{0}{0}$.

$\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ and $f(c) = 0$, $g(c) = 0$.

a. What property must the functions $f(x)$ and $g(x)$ have at the point $(c, 0)$? ____________________

____

b. Write the equation of the tangent lines of both functions at the point $(c, 0)$. ____________________

c. Substitute the linearization for each of the functions near the point $(c, 0)$.
Then simplify the resulting ratio.

$\frac{f(x)}{g(x)} \approx$ ____________ = ____________________

> **NOTE**
>
> It is important that you use the approximation symbol since a linearization only approximates a function at points "very near" a point of tangency. Remember that the exact meaning of "very near" is what Leibniz and Newton understood but could not explain. They could not show what they meant by graphing and Zooming In. The notion of a limit was ahead of its time.

d. According to your work in #7 a, b, and c, $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ = ____________.

What limit must exist in order for this to be true? ____________

Johann Bernoulli fought for his position among the great mathematicians all of his life. His older brother, James or Jacob, was his tutor and claimed credit for little brother's accomplishments. When John wanted to secure his first job, big brother already held the most prestigious position. Later, John had a son Daniel who also excelled in mathematics. Once they both competed for the same prize and Daniel won. John's response was to kick Daniel out of the house.

This controversial man was one of eight in the Bernoulli family who were important mathematicians. Johann was a teacher of great mathematicians and the author of thousands of letters containing mathematical instruction. He played a significant role in the development of calculus; however he was often contentious with his peers over the authorship of new ideas. It is ironic that the **Rule for the Indeterminate Form 0/0**, which was actually discovered by Johann Bernoulli, bears not his name but the name of his student, the man who recorded it in a textbook, **Marquis de l'Hopital**.

L'Hopital's Rule

Let f and g be differentiable functions, so that

a) as $x \to a$, either $f(x) = 0$ and $g(x) = 0$ or

$f(x) \to \pm\infty$ and $g(x) \to \pm\infty$;

and

b) $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exists,

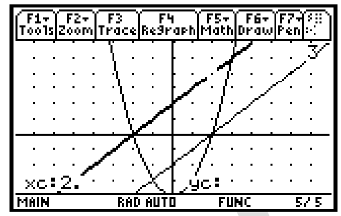
then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$.

Note: The parameter a may be equal $\pm\infty$ and the limits may be one-sided.

Limits Revisited: *L'Hopital's Rule*

8. $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} =$ ______

Follow the steps below to determine the value of the limit.



a. Calculate the value of the limit on the Home screen.
Write your answer to #8 on the line above.

- Let y1(x) = $f(x) = x^2 - 4$
- Let y2(x) = $g(x) = x - 2$
- Let y3(x) = $h(g) = \frac{y1(x)}{y2(x)}$.
- Use [2nd] [F1] for [F6] **Style**: [4] **Thick** for y3(x).
- [F2] Zoom [F4] **ZoomDec**: [◆] [F2] for [WINDOW] to adjust **ymax** = 6, **xres** = 3: [◆] [F3] for [GRAPH].

b. Does the limit above satisfy part (a) of the hypothesis of l'Hopital's Rule? Explain. ________________________________

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c. Does the limit above satisfy part (b) of the hypothesis of l'Hopital's Rule? Explain.________________________________

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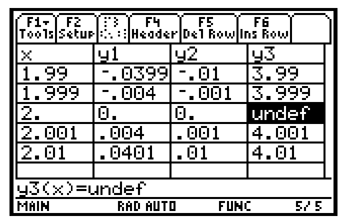
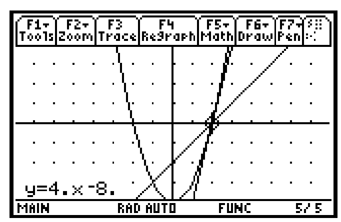


d. Support your answer numerically as follows:
- [◆] [F4] for [TblSet]: Independent–**ASK**: [ENTER]
- [◆] [F5] for [TABLE] [ENTER]:
- Enter the x-values: 1.99 – 1.999 – 2 – 2.001 – 2.01
- Your table should look like the one at the right.
- Complete the sentences below:
  1. As x approaches 2, $f(x)$ approaches ____________.
  2. As x approaches 2, $g(x)$ approaches ____________.
  3. As x approaches 2, $h(x)$ approaches ____________.



e. Support l'Hopital's Rule graphically.
- Graph y1(x) = $f(x)$ and y2(x) = $g(x)$.
- Use [F2] Zoom [4] **ZoomDec**
- Use [F5] Math **A: Tangent**
- Then enter the x-coordinate [2] [ENTER] to see the tangent line of $f(x)$ at the point (2, 0).
- Explain how this graph illustrates the meaning of l'Hopital's Rule.

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Name:

CHAPTER 10

Lesson 45

Limits Revisited: *L'Hopital's Rule*

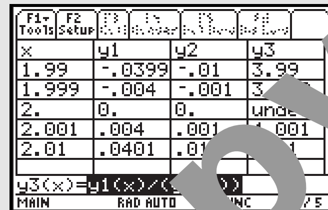
Journal Entries

1. Write l'Hopital's Rule in your Journal.

2. Define and write all of the indeterminate forms.

True or False; and fill in the blank.

3. _______ This rule says that when a limit has the form $\frac{0}{0}$, take the derivative of the limit.

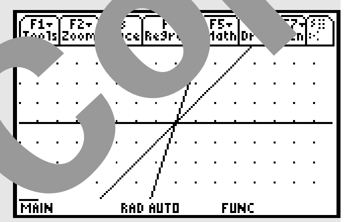


4. _______ An example of this form is $\lim_{x \to 0} \frac{x^3 - 8}{x - 2}$.

5. _______ In order to use this rule, x must approach zero.

6. _______ This rule says that when a limit has the form $\frac{0}{0}$, take the ratio of the derivatives.

7. _______ This rule says $\frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$.



8. _______ When l'Hopital's Rule applies, the ratio that their y-values approach zero is determined by the rates that the functions approach zero.

9. This rule was discovered by ______________________________ in 1694.

10. This rule was named for ________________________ who included it in the first calculus textbook in 1696.